NUMERICAL MODELING OF TURBULENT-TRANSFER PROCESSES IN A MIXING ZONE

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UDC 532.517.4

The series of moments of the velocity field in a two-dimensional zone of mixing is calculated in this article by numerically solving a system of turbulent-transfer differential equations derived from an equation for a single-point distribution function of the velocity pulsation field [1] and simplified to an approximation of the boundary layer. The closed form of the transfer equation is obtained at the level of the third moments using the Millionshchikov hypothesis [2]. The differential operator of the system under this closure turns out to be weakly hyperbolic [3], and not parabolic. A difference scheme is proposed that realizes the method of matrix fitting [4]. A comparison is carried out with an experiment [5, 6].

1. Turbulent Transfer Differential Equations in a Two-Dimensional Zone of Mixing

A model of turbulent transfer has been described [1] in which equations were derived for the mean values and second moments of the velocity field from an equation for a single-point distribution function for the velocity-field pulsations. Equations for moments of higher orders can be obtained analogously. Equations will be written out below for the moments of the first three orders of the velocity field for a nonhomogeneous turbulent flow, which are referred to the developed turbulent flow of a completely turbulent fluid [1] under steady-state external conditions in the absence of a pressure gradient.

The continuity equation is given by

$$\frac{\partial \langle u_k \rangle}{\partial x_k} = 0. \tag{1.1}$$

The equation for conservation of momentum for the mean velocity is given by

$$\frac{\partial}{\partial x_k} \langle u_k \rangle \langle u_\alpha \rangle + \frac{\partial}{\partial x_k} \langle u_k' u_\alpha' \rangle = 0. \tag{1.2}$$

The equation for the tensor components of the turbulent stresses is given by

$$\frac{\partial}{\partial x_k} \left[\langle u_k \rangle \langle u_\alpha' u_\beta' \rangle + \langle u_k' u_\alpha' u_\beta' \rangle \right] + \langle u_\beta' u_k' \rangle \frac{\partial \langle u_\alpha \rangle}{\partial x_k} + \langle u_\alpha' u_k' \rangle \frac{\partial \langle u_\beta \rangle}{\partial x_k} = \tag{1.3}$$

$$= -\frac{3}{4} \cdot \frac{a_0}{L} \cdot E^{1/2} \langle u_\alpha' u_\beta' \rangle + \frac{b_0}{L} \cdot E^{1/2} \cdot \left[\frac{2}{3} E \delta_{\alpha\beta} - \langle u_\alpha' u_\beta' \rangle \right].$$

The equation for the third moments of the velocity-field pulsations* is given by

*Equations (1.1)-(1.3) were presented in [1].

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 125-133, May-June, 1975. Original article submitted May 23, 1974.

$$\langle u_k\rangle \frac{\partial}{\partial x_k}\langle u'_\alpha u'_\beta u'_\gamma\rangle = -\left\{\langle u'_\beta u'_\gamma u'_k\rangle \frac{\partial \langle u_\alpha\rangle}{\partial x_k} + \langle u'_\alpha u'_\gamma u'_k\rangle \frac{\partial \langle u_\beta\rangle}{\partial x_k} + \right.$$ (1.4)

$$\left. + \langle u'_\alpha u'_\beta u'_k\rangle \frac{\partial \langle u_\gamma\rangle}{\partial x_k}\right\} + \left\{\langle u'_\alpha u'_\beta\rangle \frac{\partial \langle u'_\gamma u'_k\rangle}{\partial x_k} + \langle u'_\beta u'_\gamma\rangle \frac{\partial \langle u'_\alpha u'_k\rangle}{\partial x_k} + \right.$$

$$\left. + \langle u'_\alpha u'_\gamma\rangle \frac{\partial \langle u'_\beta u'_k\rangle}{\partial x_k}\right\} - \frac{\partial}{\partial x_k}\langle u'_\alpha u'_\beta u'_\gamma u'_k\rangle - \langle u'_\alpha u'_\beta u'_\gamma\rangle \cdot \pi^{-1} \cdot E^{1/2}.$$

The equation for the mean density of the kinetic energy for pulsating motion [obtained by summing Eq. (1.3) over $\alpha = \beta$] is given by

$$\frac{\partial}{\partial x_k}\left[\langle u_k\rangle E + \langle E' \cdot u'_k\rangle\right] + \langle u'_\beta u'_k\rangle \frac{\partial \langle u_\beta\rangle}{\partial x_k} = -\frac{3}{4} \cdot \frac{a_0}{L} E^{3/2}.$$ (1.5)

The equation for turbulence intensity along separate coordinate axes (not summed over α !) is given by

$$\frac{\partial}{\partial x_k}\left[\langle u_k\rangle E_\alpha + \langle E'_\alpha u'_k\rangle\right] + \langle u'_\alpha u'_k\rangle \frac{\partial \langle u_\alpha\rangle}{\partial x_k} = -\frac{3}{4} \cdot \frac{a_0}{L} E_\alpha E^{1/2} + \frac{b_0}{L} E^{1/2}\left[\frac{1}{3} E\delta_{\alpha\alpha} - E_\alpha\right].$$ (1.6)

In Eqs. (1.1)-(1.6) we have set

$$E = \frac{1}{2}\langle u'_k u'_k\rangle; \quad E' = \frac{1}{2} u'^2_k; \; \pi = L \cdot \left[\frac{9}{8} a_0 + b_0\right]^{-1},$$

where L is the scale of turbulence characterizing the "mean dimension" of the turbulent elements in the flow [1] and a_0 and b_0 are empirical constants.

The system of equations (1.1)-(1.6) is not closed, and the number of unknown functions is greater than the number of equations. To obtain closed equations, we use the hypothesis of fourth moments [2], according to which the fourth moments of a velocity field are expressed in terms of the second moments using equations that are true for a normal distribution,

$$\langle u'_\alpha u'_\beta u'_\gamma u'_k\rangle = \langle u'_\alpha u'_\beta\rangle\langle u'_\gamma u'_k\rangle + \langle u'_\alpha u'_\gamma\rangle\langle u'_\beta u'_k\rangle + \langle u'_\alpha u'_k\rangle\langle u'_\beta u'_\gamma\rangle.$$ (1.7)

Let us consider the two-dimensional turbulent-mixing problem of a plane homogeneous flow with stationary fluid (called the "jet edge" problem) (Fig. 1). More precisely, we will consider steady-state plane flow of boundary-layer type, in which the mean values are independent of one of the Cartesian coordinate axes and in which the component of the mean velocity along this coordinate is identically zero. The scale size of the flow region along one of the remaining coordinate axes is assumed to be much less than the scale size along the other axis, which in the case of the flow depicted in Fig. 1 is experimentally confirmed [5, 6]. Equations (1.1)-(1.6) are simplified in this sense by drawing on experimental data [5, 6]. The result of this simplification is the following system of turbulent-transfer equations:

$$\frac{\partial \langle u\rangle}{\partial x} + \frac{\partial \langle v\rangle}{\partial y} = 0; \quad \langle u\rangle \frac{\partial \langle u\rangle}{\partial x} + \langle v\rangle \frac{\partial \langle u\rangle}{\partial y} + \frac{\partial \langle u'v'\rangle}{\partial y} = 0;$$ (1.8)

$$\langle u\rangle \frac{\partial \langle u'v'\rangle}{\partial x} + \langle v\rangle \frac{\partial \langle u'v'\rangle}{\partial y} + \frac{\partial \langle u'v'^2\rangle}{\partial y} + \langle v'^2\rangle \frac{\partial \langle u\rangle}{\partial y} = -\left[\frac{3}{4} \cdot \frac{a_0}{L} + \frac{b_0}{L}\right] E^{1/2} \langle u'v'\rangle;$$

$$\langle u\rangle \frac{\partial E}{\partial x} + \langle v\rangle \frac{\partial E}{\partial y} + \langle u'v'\rangle \frac{\partial \langle u\rangle}{\partial y} + \frac{\partial \langle E' \cdot v'\rangle}{\partial y} = -\frac{3}{4} \cdot \frac{a_0}{L} E^{3/2};$$

$$\langle u\rangle \frac{\partial \langle v'^2\rangle}{\partial x} + \langle v\rangle \frac{\partial \langle v'^2\rangle}{\partial y} + \frac{\partial \langle v'^3\rangle}{\partial y} = -\frac{3}{4}\frac{a_0}{L} E^{1/2} \langle v'^2\rangle - \frac{b_0}{L} E^{1/2}\left[\langle v'^2\rangle - \frac{2}{3} E\right];$$

$$\langle u\rangle \frac{\partial \langle u'v'^2\rangle}{\partial x} + \langle v\rangle \frac{\partial \langle u'v'^2\rangle}{\partial y} + \langle v'^3\rangle \frac{\partial \langle u\rangle}{\partial y} + \langle u'v'\rangle \frac{\partial \langle v'^2\rangle}{\partial y} + 2\langle v'^2\rangle \frac{\partial \langle u'v'\rangle}{\partial y} = -\pi^{-1} E^{1/2} \langle u'v'^2\rangle;$$

$$\langle u\rangle \frac{\partial \langle v'^3\rangle}{\partial x} + \langle v\rangle \frac{\partial \langle v'^3\rangle}{\partial y} + 3\langle v'^2\rangle \frac{\partial \langle v'^2\rangle}{\partial y} = -\pi^{-1} E^{1/2} \langle v'^3\rangle;$$

$$\langle u\rangle \frac{\partial \langle E' \cdot v'\rangle}{\partial x} + \langle v\rangle \frac{\partial \langle E' \cdot v'\rangle}{\partial y} + \langle u'v'^2\rangle \frac{\partial \langle u\rangle}{\partial y} + \langle v'^2\rangle \frac{\partial E}{\partial y} + \langle u'v'\rangle \frac{\partial \langle u'v'\rangle}{\partial y} + \langle v'^2\rangle \frac{\partial \langle v'^2\rangle}{\partial y} = -\pi^{-1} E^{1/2} \langle E' \cdot v'\rangle.$$

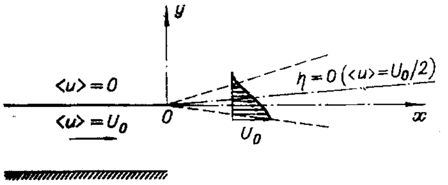

Fig. 1

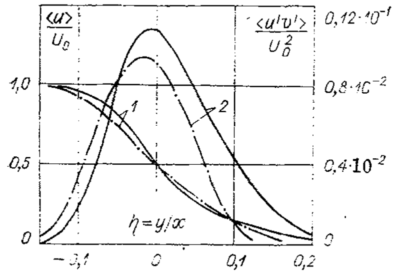

Fig. 2

The third moments are (approximately) expressed from the last three equations of the system (1.8) in terms of the second moments by means of the equations

$$-\langle u'v'^2\rangle \approx \pi E^{-1/2}\left[\langle u'v'\rangle \frac{\partial\langle v'^2\rangle}{\partial y} + 2\langle v'^2\rangle \frac{\partial\langle u'v'\rangle}{\partial y}\right]; \tag{1.9}$$

$$-\langle v'^3\rangle \approx \pi E^{-1/2}\left[3\langle v'^2\rangle \frac{\partial\langle v'^2\rangle}{\partial y}\right];$$

$$-\langle E'\cdot v'\rangle \approx \pi E^{-1/2}\langle v'^2\rangle \frac{\partial E}{\partial y}.$$

After substituting Eqs. (1.9) in Eqs. (1.8) we obtain a system of five equations for determining the mean values and second moments of the velocity-field pulsations,

$$\frac{\partial\langle u\rangle}{\partial x} + \frac{\partial\langle v\rangle}{\partial y} = 0;\quad \langle u\rangle\frac{\partial\langle u\rangle}{\partial x} + \langle v\rangle\frac{\partial\langle u\rangle}{\partial y} + \frac{\partial\langle u'v'\rangle}{\partial y} = 0; \tag{1.10}$$

$$\langle u\rangle\frac{\partial\langle u'v'\rangle}{\partial x} + \langle v\rangle\frac{\partial\langle u'v'\rangle}{\partial y} + \langle v'^2\rangle\frac{\partial\langle u\rangle}{\partial y} = \pi\frac{\partial}{\partial y}\left[E^{-1/2}\times\right.$$

$$\left.\times\left(\langle u'v'\rangle\frac{\partial\langle v'^2\rangle}{\partial y} + 2\langle v'^2\rangle\frac{\partial\langle u'v'\rangle}{\partial y}\right)\right] - \langle u'v'\rangle E^{1/2}\cdot\left[\frac{3}{4}\cdot\frac{a_0}{L} + \frac{b_0}{L}\right];$$

$$\langle u\rangle\frac{\partial E}{\partial x} + \langle v\rangle\frac{\partial E}{\partial y} + \langle u'v'\rangle\frac{\partial\langle u\rangle}{\partial y} = \pi\frac{\partial}{\partial y}\left[E^{-1/2}\cdot\langle v'^2\rangle\frac{\partial E}{\partial y}\right] - \frac{3}{4}\cdot\frac{a_0}{L}E^{3/2};$$

$$\langle u\rangle\frac{\partial\langle v'^2\rangle}{\partial x} + \langle v\rangle\frac{\partial\langle v'^2\rangle}{\partial y} - 2\langle v'^2\rangle\frac{\partial\langle u\rangle}{\partial x} = \pi\frac{\partial}{\partial y}\left[3E^{-1/2}\langle v'^2\rangle\times\right.$$

$$\left.\times\frac{\partial\langle v'^2\rangle}{\partial y}\right] - \frac{3}{4}\cdot\frac{a_0}{L}\langle v'^2\rangle E^{1/2} - \frac{b_0}{L}E^{1/2}\left[\langle v'^2\rangle - \frac{2}{3}E\right].$$

The system (1.10) is written in dimensionless form. The velocity of the mainstream flow U_0 is taken as the scale size of the velocity (of Fig. 1). No linear scale size exists for the zone of mixing. We should therefore expect the developed turbulent structure of this flow to be self-consistent (this is confirmed by experiment [5, 6]). The spatial variables x and y in Eqs. (1.10) are dimensionlessly expressed by means of an arbitrary linear scale for the (numerical) solution. The numerical results are represented in terms of the self-consistent variable η = x/y for comparison with the experimental data.

The equation for the turbulence scale L is derived on the basis of concepts of dimensionality and similitude and agrees with the experimentally discovered [5, 6] self-consistent flow structure,

$$L = \alpha x, \tag{1.11}$$

where α is a proportionality factor. The turbulence scale determined by Eq. (1.11) has the same functional form as the integral turbulence scale. However, the turbulence scale L ("mean dimension" of an element of fluid in the flow) is not equal to the integral scale, but is proportional to it. If the integral turbulence scale can be found from (longitudinal and transverse) correlation functions measured in an experiment [6], the scale L, which is an integral of the spectral distribution of turbulent kinetic energy over the entire space of wave numbers [1, 7], is not experimentally determined. Consequently, the value of the proportionality factor α also cannot be found from the experimental data. The ratio of the empirical constants a_0 and b_0 to the turbulence scale L everywhere occurs in the right sides of Eqs. (1.10). This permits the ratio a_0 /L(a_0/α) to be found by comparing a theoretical approximation for the dissipation of turbulent kinetic energy, having the form

$$\varepsilon = xU_0^{-3}\frac{3}{4}\frac{a_0}{L}E^{3/2}, \tag{1.12}$$

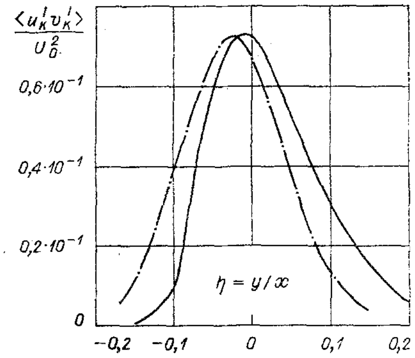

Fig. 3

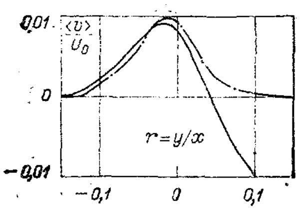

Fig. 4

to the experimental values for this magnitude obtained by adding the mean squares of the derivatives, which determine the dissipation ε. Some of these derivatives were measured in an experiment [6], while the others were expressed in terms of measured values by drawing on a number of hypotheses. The equation for the dissipation rate (1.12) satisfactorily approximates the experimental distribution of the variable ε when the ratio $(a_0/\alpha) \cong 6$ is constant. The ratio of the dissipative and exchange constants $(3/4)a_0/b_0$ (in numerical calculations) was taken equal to one.

We may complement the system (1.10) with a differential equation for L. However, the number of "empirical constants" grows, at least to six, only a small number of them being "taken from experiment." The remaining constants require for their determination a somewhat significant computational optimization [8, 9]. Current in the zone of mixing is formed by an external homogeneous flow moving with constant velocity U_0, so that at "negative infinity" five conditions are set for the five desired functions defined by the system (1.10):

$$\text{as} \quad y \to -\infty : \langle u \rangle \to 1;\ \langle u'v' \rangle \to 0;\ E \to E_-;\ \langle v'^2 \rangle \to \langle v'^2 \rangle_-;\ \langle v \rangle \to 0. \tag{1.13}$$

At the external boundary of the zone of mixing, adjacent to the quiescent fluid, the following physically obvious boundary conditions are homogeneous:

$$\text{as} \quad y \to +\infty : \langle u \rangle = \langle u'v' \rangle = E = \langle v'^2 \rangle = 0, \tag{1.14}$$

while the condition on the function $\langle v'^2 \rangle$ on the external boundary is a consequence of the condition on the function E. The condition $\langle v(x, -\infty) \rangle = 0$ for the zone of mixing is not completely obvious and requires explanation. The pattern of the mixing of a homogeneous semi-infinite flow with quiescent fluid (cf. Fig. 1) was idealized. In fact, the zone of mixing is always formed at the boundary of a jet of finite width with a surrounding stationary fluid. We may therefore assume that the transverse velocity $\langle v \rangle$ vanishes along a streamline near the axial jet line, i.e., this streamline (and, consequently, the characteristic $k_2 \equiv dx : dy = \langle u \rangle / \langle v \rangle$) is a straight line.

The conditions of Eqs. (1.13) and (1.14) are set for a "sufficiently large" distance from both boundaries of the zone of mixing, in which the desired functions begin to vary in absolute value within a preassigned accuracy. All five desired functions must be defined in the initial section at $x = x_0$. The functions $\langle u \rangle$, E, and $\langle v'^2 \rangle$ in the section $x = x_0$ were defined (in a numerical calculation) in the form of "smoothened" step functions, and $\langle u'v' \rangle$, in the form of a "smoothened" column function. The function $\langle v(x_0, y) \rangle$ in the initial section of $x = x_0$ is set equal to zero. This condition is approximate (due to the smoothness of $\langle u(x_0, y) \rangle$) which is not of importance in view of the expected self-similarity of the mean characteristics of the velocity field in the zone of mixing. A specification of finite value for the functions E and $\langle v'^2 \rangle$ at negative infinity corresponds to a small initial turbulence level, which always occurs in an external flow. (The values $E = 10^{-8}$ and $\langle v'^2 \rangle = 2/3\ E$ — were taken in the numerical calculations.)

2. Difference Scheme and Its Numerical Realization

We make several remarks on the nature of the system of differential transfer equations (1.10). The equation for conservation of momentum [second equation of (1.10)] is not a diffusion-type equation and the

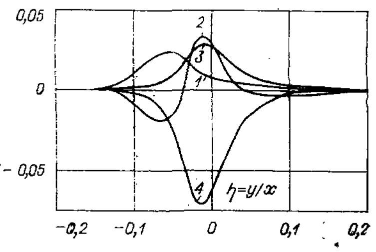

Fig. 5

system (1.10) is itself not of diffusion type. We may verify, operating in the usual way [10], that the system (1.10) has, in addition to trivial septuple characteristics x = const ($k_1 \equiv dx : dy = 0$), one more real characteristic coinciding with the streamline $k_2 \equiv dx : dy = \langle u \rangle / \langle v \rangle$. Thus the system (1.10) has real multiple characteristics. The differential operator of Eqs. (1.10) can therefore be seen to be of weakly hyperbolic type [3]. We recall that the initial system of differential equations (1.1)-(1.6) was simplified in approximation to a boundary layer and was then closed at the level of the third moments. This closure also led to a weakly hyperbolic type, and not to the more usual parabolic differential operator for turbulent-transfer equations. We may note in passing that Bradshaw, Ferris, and Atwell [11] numerically solved a system of three first-order equations which also turned out to be not parabolic, but strictly hyperbolic [10], in calculating the mean characteristics of the velocity field in a turbulent boundary layer. One characteristic was vertical, as in any boundary-layer-type system, for this system of equations.

The difference scheme* for Eqs. (1.10) was constructed on the basis of principles set forth by Godunov [12] and has the form

when $v_n^m < 0$.

$$u_n^m \frac{u_n^{m+1} - u_n^m}{\Delta x} + v_n^m \frac{u_{n+1}^{m+1} - u_n^{m+1}}{\Delta y} + \frac{\tau_{n+1}^{m+1} - \tau_{n-1}^{m+1}}{2\Delta y} = 0; \tag{2.1}$$

$$u_n^m \frac{\tau_n^{m+1} - \tau_n^m}{\Delta x} + v_n^m \frac{\tau_{n+1}^{m+1} - \tau_n^{m+1}}{\Delta y} + \varphi_n^m \frac{u_n^{m+1} - u_{n-1}^{m+1}}{\Delta y} - \frac{\tau_{n+1}^{m+1} - 2\tau_n^{m+1} + \tau_{n-1}^{m+1}}{2\Delta y} = f_1;$$

$$u_n^m e_n^m \frac{e_n^{m+1} - e_n^m}{\Delta x} + v_n^m e_n^m \frac{e_{n+1}^{m+1} - e_n^{m+1}}{\Delta y} + \frac{\tau_n^m}{2} \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2\Delta y} = f_2$$

$$u_n^m \frac{\varphi_n^{m+1} - \varphi_n^m}{\Delta x} + v_n^m \frac{\varphi_{n+1}^{m+1} - \varphi_n^{m+1}}{\Delta y} - 2\varphi_n^m \frac{u_n^{m+1} - u_n^m}{\Delta x} - \frac{\tau_n^m}{2} \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{2\Delta y} = f_3.$$

When $v_n^m \geqslant 0$

$$u_n^m \frac{u_n^{m+1} - u_n^m}{\Delta x} + v_n^m \frac{u_n^{m+1} - u_{n-1}^{m+1}}{\Delta y} + \frac{\tau_{n+1}^{m+1} - \tau_{n-1}^{m+1}}{2\Delta y} = 0; \tag{2.2}$$

$$u_n^m \frac{\tau_n^{m+1} - \tau_n^m}{\Delta x} + v_n^m \frac{\tau_n^{m+1} - \tau_{n-1}^{m+1}}{\Delta y} + \varphi_n^m \frac{u_n^{m+1} - u_{n-1}^{m+1}}{\Delta y} - \frac{\tau_{n+1}^{m+1} - 2\tau_n^{m+1} + \tau_{n-1}^{m+1}}{2\Delta y} = f_1;$$

$$u_n^m e_n^m \frac{e_n^{m+1} - e_n^m}{\Delta x} + v_n^m e_n^m \frac{e_n^{m+1} - e_{n-1}^{m+1}}{\Delta y} + \frac{\tau_n^m}{2} \cdot \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2 \cdot \Delta y} = f_2,$$

$$u_n^m \frac{\varphi_n^{m+1} - \varphi_n^m}{\Delta x} + v_n^m \frac{\varphi_n^{m+1} - \varphi_{n-1}^{m+1}}{\Delta y} - 2\varphi_n^m \frac{u_n^{m+1} - u_n^m}{\Delta x} - \frac{\tau_n^m}{2} \cdot \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{2\Delta y} = f_3.$$

The averaging signs have been omitted in Eqs. (2.1) and (2.2) for all values and we have set $\tau \equiv \langle u'v' \rangle$; $\varphi \equiv \langle v'^2 \rangle$; $e \equiv E^{1/2}$, in these equations,

*The system (1.10) is represented in the difference form (2.1), (2.2) without a continuity equation. We will speak below of its inclusion in a difference algorithm.

$$f_1 = \frac{\pi_n^m}{(\Delta y)^2}\left\{\left(\frac{\tau}{e}\right)^m_{n+\frac{1}{2}}\varphi^{m+1}_{n+1} - \left[\left(\frac{\tau}{e}\right)^m_{n+\frac{1}{2}} + \left(\frac{\tau}{e}\right)^m_{n-\frac{1}{2}}\right]\varphi^{m+1}_n +\right.$$

$$+\left(\frac{\tau}{e}\right)^m_{n-\frac{1}{2}}\varphi^{m+1}_{n-1} + \left(\frac{\varphi}{e}\right)^m_{n+\frac{1}{2}}\tau^{m+1}_{n+1} - \left[\left(\frac{\varphi}{e}\right)^m_{n+\frac{1}{2}} + \left(\frac{\varphi}{e}\right)^m_{n-\frac{1}{2}}\right]\tau^{m+1}_n +$$

$$\left.+\left(\frac{\varphi}{e}\right)^m_{n-\frac{1}{2}}\tau^{m+1}_{n-1}\right\} + \frac{1}{2}\left[\left(\frac{3}{4}a_0 + b_0\right)/L^m_n\right]\left[e^{m+1}_n\cdot\tau^m_n + e^m_n\cdot\tau^{m+1}_n\right];$$

$$f_2 = \frac{\pi_n^m}{(\Delta y)^2}\left\{\varphi^m_{n+\frac{1}{2}}\cdot e^{m+1}_{n+1} - \left[\varphi^m_{n+\frac{1}{2}} + \varphi^m_{n-\frac{1}{2}}\right]e^{m+1}_n + \varphi^m_{n-\frac{1}{2}}e^{m+1}_{n-1}\right\} - \frac{3}{8}\cdot\frac{a_0}{L^m_n}\left[(e^3)^m_n + 3(e^2)^m_n\cdot(e^{m+1}_n - e^m_n)\right];$$

$$f_3 = \frac{\pi_n^m}{(\Delta y)^2}\left\{3\left(\frac{\varphi}{e}\right)^m_{n+\frac{1}{2}}\varphi^{m+1}_{n+1} - 3\left[\left(\frac{\varphi}{e}\right)^m_{n+\frac{1}{2}} + \left(\frac{\varphi}{e}\right)^m_{n-\frac{1}{2}}\right]\varphi^{m+1}_n +\right.$$

$$\left.+3\left(\frac{\varphi}{e}\right)^m_{n-\frac{1}{2}}\varphi^{m+1}_{n-1}\right\} - \frac{1}{2}\left[\left(\frac{3}{4}a_0 + b_0\right)/L^m_n\right]\left(e^{m+1}_n\varphi^m_n + e^m_n\varphi^{m+1}_n\right) +$$

$$+\frac{2}{3}b_0/L^m_n\left[(e^3)^m_n + 3(e^2)^m_n(e^{m+1}_n - e^m_n)\right].$$

The difference scheme (2.1), (2.2) approximates the system of differential equations (1.10) (without a continuity equation) to the first order of accuracy. No special investigation into the stability of the difference scheme was attempted. The scheme is implicit, and its absolute computational stability was experimentally confirmed by carrying out computations for different ratios of steps $\Delta x/\Delta y$. The calculation was always stable. The system of differential equations (2.1), (2.2) can be written in the form of the single matrix equation

$$A_n\vec{X}_{n+1} + B_n\vec{X}_n + C_n\vec{X}_{n-1} = \vec{D}_n, \qquad (2.3)$$

where A, B, and C are fourth-order square matrices, $\vec{D}_n$ is the vector of the right sides, and $\vec{X} = [u, \tau, e, \varphi]$ is the desired vector. The equations for A, B, C, and D_n are constructed from Eqs. (2.1) and (2.2) and will not be presented here. A matrix fitting method [4] discussed in detail in [13] was used to solve Eq. (2.3).

The transverse component of the mean flow velocity in the (m + 1)-th layer was calculated using an equation obtained from the continuity equation, that is,

$$\langle v\rangle = -\int_{-\infty}^{y}\frac{\partial\langle u\rangle}{\partial x}dy. \qquad (2.4)$$

Calculation of the velocity $\langle v\rangle$ can be directly included in the difference algorithm. However, this increases the order of the matrix to be transformed at each node of the difference network [then solving the matrix equation (2.3) by means of fitting] by one and, consequently, increases computation time. A calculation of the transverse velocity was carried out using Eqs. (2.4).

3. Numerical Results. Comparison with Experiment

The flow field in the (x, y) plane was covered by a difference network with steps Δx and Δy in the directions of the x and y axes, respectively, in order to numerically solve Eq. (2.3). The number of nodes along the y axis was increased in the course of computation as a consequence of the expansion of the zone of mixing with increasing x coordinate (cf. Fig. 1). The sizes of the Δx and Δy steps were selected to ensure that the precision of the solution of the problem will be preserved. The matrix fitting algorithm for Eqs. (2.3) was realized in the form of a program written in ALGOL for the ALPHA-6 translator of the Computer Center, Siberian Division, Academy of Sciences of the USSR (using the BÉSM-6 computer). On the average, $1.0\cdot10^3$ steps along the x coordinate were required to attain a self-similar region.

Numerical results are presented in the form of curves in Figs. 2-5. Profiles of the numerical solution are indicated in all the figures by solid lines and experimental profiles [6] by dot-dashed lines. In

Fig. 2, the digit 1 denotes the mean velocity profile $\langle u \rangle$ and the digit 2, the stress of the turbulent friction $\langle u'v' \rangle$. The difference in the behavior of the velocity profiles is also characteristic for the experimental profiles [5, 6] (cf. Fig. 7 in [14], in which experimental profiles of these two works reveal as distinct a behavior as in Fig. 2). The velocity profile is nearer previous data [5] on the internal side of the zone of mixing ($\eta < 0$). The profile of the turbulent stresses of the numerical solution are shifted in the direction of the external side of the zone ($\eta > 0$) and differs there most from the experimental solution. Figure 3 depicts curves for the total turbulence intensity. The correspondence between the calculated profile $\langle u'_k u'_k \rangle$ and the experimental profile is entirely satisfactory, though here, as in the case of turbulent stress, a shift towards the zone of mixing adjacent to the quiescent fluid is revealed. The transverse flow velocity component $\langle v \rangle$ is depicted in Fig. 4. Separate components of the total turbulent kinetic energy balance (twice its value) are plotted in Fig. 5. The digit 1 refers to convection, 2, to diffusion, 3, to dissipation, and 4, to generation of turbulent energy. All these curves reveal a qualitative and quantitative agreement to data from both [6] and [5] (cf. a similar graph in [15]). The calculated curve for the function $\langle v'^2 \rangle$ reveals a qualitatively correct behavior, but with a noticeable shift toward the external side of the zone of mixing. Its maximal value varies by 20% less than in [5], which varied by 25% less than in another experiment [6]. The difference in the intensities of the velocity-field fluctuations measured in [5, 6] is explained by the different "initial conditions" of the experimental apparatus in these experiments. The influence of initial conditions on the statistical properties of turbulence in a zone of mixing has undergone experimental study [14]. The general conclusion is that these contributions affect in a complex way the adaptation of a displacing layer to a self-similar state and that the value of the intensities of the velocity fluctuations are functions of the level of disturbances in a boundary layer in a Washburn riser (cf. Fig. 1).

The author wishes to express his appreciation to S. K. Godunov for consultation in carrying out the construction of the difference scheme, and to A. T. Onufriev for discussion of problems in turbulent transfer.

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